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Z TRANSFORM

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ABSTRACT

An introduction to Z Transform is the topic of this paper. It deals with a review of what Z Transform means and what does the specific region of convergence represent. A pictorial representation of the region of convergence has been sketched. Most of the important Z Transform properties have been mentioned with detailed proofs. This paper also insights the Inverse Z Transform and provides multiple methods to find the same. The use of Z Transform is to convert a simple or a complicated sequence in to a corresponding frequency domain equivalent. A brief description of the same has also been given.

KEYWORDS: Z Transform, Inverse Z Transform, Convolution, Region of Convergence

1. INTRODUCTION

This paper deals with the general introduction to what Z Transform is and what is it used for. Mathematical studies have got a number of transform such as the Laplace Transform, the Fourier Transform, etc. But, both Laplace and Fourier Transforms are continuous functions and cannot be used to study discrete systems. There is a way to convert the continuous Fourier Transform into its discrete equivalent by finding the Discrete Fourier Transform (DFT) but Fourier Transform itself cannot be used to study discrete systems since it is continuous. Linear systems in which the input signals are in the form of discrete pulses are called as ‘Linear Time Variants’. For the analysis of such systems, we need Z Transform. The Z Transform will convert a sequence into a corresponding frequency domain equivalent.

2. SEQUENCES

Z transform is always defined for a specific sequence. If ‘n’ items are arranged according to a certain rule, then this arrangement will be known as a sequence. Thus, in general, an ordered set of real or complex numbers is called a sequence.

In this paper, a general sequence will be represented by {f(k)} with k being its index and f(k) being the k^{th} term.

3. Z TRANSFORM

3.1 Definition

Let {f(k)} = {…f(-2), f(-1), f(0), f(1), f(2)…} be the sequence of terms with index k varying form $-\infty$ to ∞.

Let $z = x + iy$ be any complex number. The Z Transform of the sequence {f(k)} is defined as,

$$Z\{f(k)\} = \cdots f(-2)z^2 + f(-1)z^1 + f(0)z^0 + f(1)z^{-1} + f(2)z^2 + \cdots$$

$$= \sum_{k=-\infty}^{\infty} f(k)z^{-k}$$

This is what represents the Z Transform of the sequence. Thus,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)z^{-k} \qquad \ldots(1)$$

4. REGION OF CONVERGENCE

Every Z Transform is defined over a region. This region is known as the region of convergence. Consider the following cases of region of convergence is

- $|z| < a$

Since z is any complex number $x + iy$, $|z| = \sqrt{x^2 + y^2}$.

Now if $|z| < a$, then, $\sqrt{x^2 + y^2} < a$

$\therefore x^2 + y^2 < a^2$

The equation $x^2 + y^2 = a^2$ is the equation of a standard circle. Thus, $x^2 + y^2 < a^2$ will represent the region inside the circle of radius a.

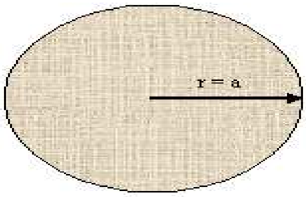

Figure 1: $|z| < a$

- $|z| > a$

Since z is any complex number $x + iy$, $|z| = \sqrt{x^2 + y^2}$.

Now if $|z| > a$, then, $\sqrt{x^2 + y^2} > a$

$\therefore x^2 + y^2 > a^2$

The equation $x^2 + y^2 = a^2$ is the equation of a standard circle. Thus, $x^2 + y^2 > a^2$ will represent the region outside the circle of radius a.

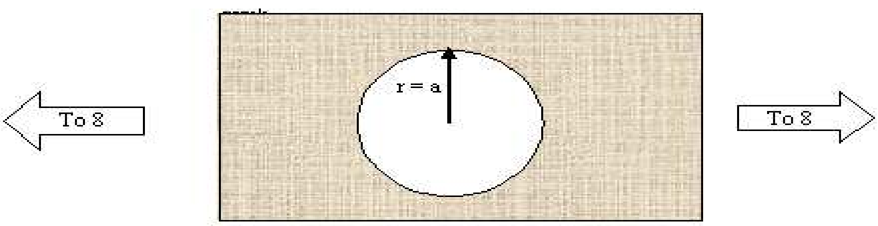

Figure 2: $|z| > a$

5. PROPERTIES OF Z TRANSFORM

5.1 Linearity Property

If p and q are constants and f(k) and g(k) are two sequences which can be added, then,

$$Z[p\ f(k) + q\ g(k)] = p\ Z[f(k)] + q\ Z[g(k)] \qquad \dots(2)$$

Proof: By definition,

$$Z[p\ f(k) + q\ g(k)] = \sum_{k=-\infty}^{\infty} [p\ f(k) + q\ g(k)]\ z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} [p\ f(k)\ z^{-k} + q\ g(k)\ z^{-k}]$$

$$= p \sum_{k=-\infty}^{\infty} f(k) z^{-k} + q \sum_{k=-\infty}^{\infty} g(k) z^{-k}$$

$$= p\ Z[f(k)] + q\ Z[g(k)]$$

This means that the Z transform of the addition or difference of two sequences is the addition or difference of the Z Transform of the two sequences itself. This holds true even when the sequence is multiplied by a real and linear constant.

5.2 Change of Scale

Multiplication or division of a variable is known as scaling. If $Z[f(k)]$ is F(z), then $Z[\ a^k f(k)] = F\left(\frac{z}{a}\right)$ and if region of convergence of $Z[f(k)]$ is $R1 < |z| < R2$ then, region of convergence of $Z[\ a^k f(k)]$ is $|a|\ R1 < |z| < |a|\ R2$.

Proof: By definition,

$$F(z) = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

Replacing z by z/a,

$$F\left(\frac{z}{a}\right) = \sum_{k=-\infty}^{\infty} f(k) \left(\frac{z}{a}\right)^{-k}$$

$$= \sum_{k=-\infty}^{\infty} a^k f(k) z^{-k}$$

$$= Z[\ a^k f(k)\ ] \qquad \dots(3)$$

If region of convergence of $Z[f(k)]$ is $R1 < |z| < R2$ then, region of convergence of $Z[\ a^k f(k)]$ is $R1 < \left|\frac{z}{a}\right| < R2$ i.e, the region of convergence will eventually be, $|a|\ R1 < |z| < |a|\ R2$

5.3 Shifting Theorem

The addition or subtraction of a variable is known as shifting. If $Z[f(k)] = F(z)$, then, and $Z[f(k-n)] = z^{-n} F(z)$

Proof: By definition,

$$Z[f(k)] = F(z) = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$\therefore Z[f(k+n)] = \sum_{k=\ \infty}^{\infty} f(k+n) z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} f(k+n) z^{-(k+n)} z^{n}$$

$$= z^{n} \sum_{k=-\infty}^{\infty} f(k+n) z^{-(k+n)}$$

Now let k + n = m. Thus, if k=∞, then m=∞.

$$\therefore Z[f(k+n)] = z^{n} \sum_{m=-\infty}^{\infty} f(m) z^{-m} = z^{n} F(z) \qquad \dots (4.1)$$

Also

$$\therefore Z[f(k-n)] = z^{n} \sum_{m=-\infty}^{\infty} f(m) z^{-m} = z^{-n} F(z) \qquad \dots (4.2)$$

5.4 Miultiplication by k

If $Z[f(k)] = F(z)$, then,

$$Z[k\, f(k)] = -z \frac{d}{dz} F(z) \qquad \dots (5)$$

Proof: By definition,

$$Z[k\, f(k)] = \sum_{k=-\infty}^{\infty} k\, f(k) z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} k\, f(k) z^{-k-1} z$$

$$= -z \sum_{k=-\infty}^{\infty} -k\, f(k) z^{-k-1}$$

$$= -z \sum_{k=-\infty}^{\infty} f(k) \frac{d}{dz}\left(z^{-k}\right)$$

$$= -z \frac{d}{dz} \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$\therefore Z[k\, f(k)] = -z \frac{d}{dz} F(z)$$

Thus, in general,

$$Z[k^n f(k)] = \left[-z\frac{d}{dz}\right]^n F(z) \qquad ...(6)$$

It should also be noted that $\left[-z\frac{d}{dz}\right]^2 \neq z^2\frac{d^2}{dz^2}$, but is actually a repetitive application of the operator $\left[-z\frac{d}{dz}\right]$, i.e., $\left[-z\frac{d}{dz}\right]^2 = \left[-z\frac{d}{dz}\right]\left[-z\frac{d}{dz}\right]$

6. CONVOLUTIONS

If f(k) and g(k) are two sequences, then their convolution f(k) * g(k) is defined as,

$$h(k) = f(k) * g(k)$$

Where,

$$h(k) = \sum_{m=-\infty}^{\infty} f(m)g(k-m)$$

$$= \sum_{m=-\infty}^{\infty} g(m)f(k-m) \qquad ...(7)$$

$$= f(k) * g(k)$$

Theorem

If h(k) is the convolution of two sequences f(k) and g(k), then,

$$Z[h(k)] = Z[f(k)]\, Z[g(k)]$$

$$i.e. \quad H(z) = F(z)G(z) \qquad ...(8)$$

Proof

By definition,

$$H(z) = Z[h(k)]$$

$$= Z[f(k) * g(k)]$$

$$= Z\left[\sum_{m=-\infty}^{\infty} f(m)g(k-m)\right]$$

$$= \sum_{k=-\infty}^{\infty}\left[\sum_{m=-\infty}^{\infty} f(m)g(k-m)\right] z^{-k}$$

Since the power series converges absolutely, it converges uniformly also and hence we can change the order of summation.

$$\therefore H(z) = \sum_{m=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} f(m)g(k-m)\right]z^{-k}$$

$$= \sum_{m=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} f(m)g(k-m)\right]z^{-k+m-m}$$

$$= \sum_{m=-\infty}^{\infty} f(m)z^{-m} \sum_{k=-\infty}^{\infty} g(k-m)z^{-(k-m)}$$

$$= \sum_{m=-\infty}^{\infty} f(m)z^{-m} \sum_{p=-\infty}^{\infty} g(p)z^{-p} \quad where\ k-m=p$$

$$\therefore H(z) = F(z)G(z)$$

7. INVERSE Z TRANSFORM

The idea behind Inverse Z Transform is exactly reverse of that of the Z Transform. In such cases, the Z Transform is already known and the original sequence can be obtained using the Inverse Z Transform. Z Transforms which are rational functions of z i.e. of the form of $F(z) = \frac{P(z)}{Q(z)}$ with P(z) and Q(z) as algebraic polynomials in z are to be considered in the discussion further. It should be noted that to find the Inverse Z Transform, it is necessary to know the region of convergence.

8. METHODS TO FIND INVERE Z TRANSFORM

The following methods can be employed to find the Inverse Z Transform of a function in z.

8.1 Direct Division

In this method, the numerator is divided by the denominator and a power series is obtained. If $F(z) = \frac{P(z)}{Q(z)}$, P(z) is actually divided by Q(z). This method is not very efficient since it is a lengthy procedure and always demands a person to express the division in the form of an existing power series which is cumbersome and complex as the function gets more and more complicated.

8.2 Binomial Expansion

This method of finding the Inverse Z Transform is relatively advanced. To apply this method, a suitable factor is to be taken in common from the denominator depending upon the region of convergence so that the denominator is of the form 1 – r where |r|<1 and then, binomial theorem will be used.

This method can be better explained by an example.

Let $F(z) = \frac{1}{z-a}$ with region of convergence as |z| < |a|. As explained earlier, the denominator will be expressed as 1 – r where |r|<1. Now as the denominator is z-a with the region of convergence as |z| < |a|,

$$F(z) = \frac{1}{a\left(\frac{z}{a} - 1\right)}$$

$$= -\frac{1}{a}\left(1 - \frac{z}{a}\right)^{-1}$$

Thus, F(z) has now been modified to a situation wherein it is possible to apply the method of binomial expansion and by using the definition of Z Transform, it will then be possible to get back the original sequence.

Using the method of binomial expansion,

$$F(z) = -\left[\frac{1}{a} + \frac{z}{a^2} + \frac{z^2}{a^3} + \cdots\right]$$

$$= \sum_{k=-\infty}^{0} -a^{k-1} z^{-k}$$

$$Z^{-1}[F(z)] = -a^{k-1} \quad , \quad k \leq 0$$

8.3 Using Partial Fractions

If F(z) can be factorized into partial fractions be it linear, quadratic or repeated, $F(z) = \frac{P(z)}{Q(z)}$ is then expressed as the sum of such factors. The constants of partial fraction are to be found and then the method of binomial expansion can be applied to find the individual Z Transform of the terms. It should be noted that if the degree of the numerator is greater than or equal to the degree of the denominator, the function F(z) is first divided by z and this is continued until the time the degree of the numerator polynomial is lesser than the degree of the denominator polynomial.

As long as there is a linear or at least a term which can be factorised in the denominator, it is convenient enough to express it as the sum of factors using the partial fraction method described above. But is there exists say a quadratic term in the denominator which cannot be factorised at all, the following method can be used.

$$Let \quad F(z) = \frac{Mz^2 + Nz}{z^2 + pz + q}$$

$$Z[c^k \cos ak] = \frac{z^2 - cz\cos a}{z^2 - 2cz\cos a + c^2}$$

$$and \quad Z[c^k \sin ak] = \frac{cz\sin a}{z^2 - 2cz\cos a + c^2}$$

Mz^2 and Nz can be expressed as,

$$Mz^2 = M[z^2 - cz\cos a] + MCz\cos a$$

$$and\ Nz = \frac{N\,cz\sin a}{c\sin a}$$

$$\frac{Mz^2 + Nz}{z^2 + pz + q} = \frac{M(z^2 - cz\cos a)}{z^2 - 2cz\cos a + c^2} + \frac{\frac{M\,cz\cos a + N}{c\sin a}\,cz\sin a}{z^2 - 2cz\cos a + c^2} \qquad \ldots(9)$$

Where, $p = -2c\cos a$ and $q = c^2$. Thus,

$$\frac{p}{-2c} = \cos a$$

$$\therefore \left|\frac{p}{2c}\right| < 1$$

Thus, if $\left|\frac{p}{2c}\right| < 1$, equation (9) can be used.

If $\left|\frac{p}{2c}\right| < 1$, the sine and cosine functions in the above expression will now be converted to corresponding hyperbolic functions.

$$\frac{Mz^2 + Nz}{z^2 + pz + q} = \frac{M(z^2 - cz\cosh a)}{z^2 - 2cz\cosh a + c^2} + \frac{\frac{M\,cz\cosh a + N}{c\sinh a}\,cz\sinh a}{z^2 - 2cz\cosh a + c^2} \quad \ldots(10)$$

9. APPLICATIONS OF Z TRANSFORM

Z transform is put to very effective use in almost every field of Applied Sciences and Technology. The greatest advantage of Z transform is that it helps convert complex sequences into their corresponding frequency domain equivalents. The Z transform is a very effective tool in the detailed analysis discrete time signals. It is also used to determine the frequency response of these discrete time signals.

10. CONCLUSIONS

This paper thus consisted of a brief overview of what Z Transform mean and what their regions of convergence represent. Major properties of Z Transform were mentioned along with detailed proofs of the same. Z Transform is very important due to its distinct property of being able to analyse discrete signals whereas Fourier Transform and Lalplace Transform, both being continuous. It goes without saying that almost every branch under signal analysis does make effective use of Z Transform.

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